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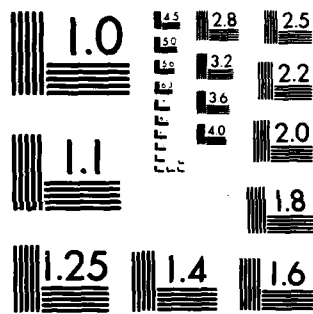
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STABILITY OF DICED SYSTEMS*

by

T. L. Johnson**

LEVEL II

ABSTRACT

Diced systems are defined as autonomous systems governed by ordinary differential equations having discontinuities (in R^n) on submanifolds where one or more of the state variables takes an integer value. Such systems may be regarded as approximations of continuous systems or as representative models of a class of discontinuous systems. Trajectories of such systems (for a given initial state) are readily calculated and may exhibit complex sliding-mode segments. Asymptotic properties of such trajectories are discussed and classified. Motivation is given in terms of observed properties of interconnected power systems.

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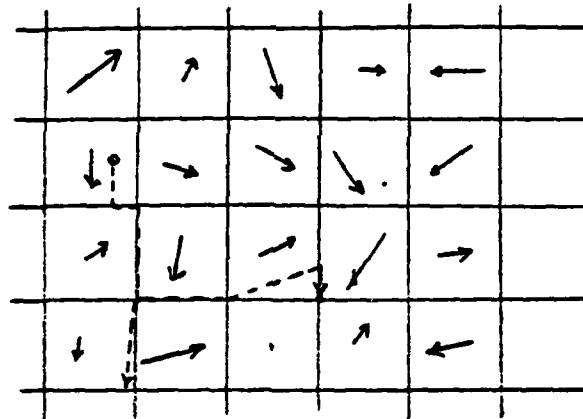
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Diced systems, as defined here, are finite-dimensional autonomous continuous-time dynamic systems governed by equations of the form $\frac{dx}{dt}(t) = f(x(t))$; $x_0(t_0) = x_0 \in \mathbb{R}^n$, $t \geq t_0$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is piecewise-constant with discontinuities only on the surfaces where one or more coordinates of \mathbb{R}^n take integer values. A diced system in \mathbb{R}^2 is very easy to illustrate: the plane can be divided into a uniform grid, and within each square a vector representing the magnitude and direction of f is shown (Figure 1).



Existence and uniqueness of a solution for any fixed initial state, x_0 , can be studied using a generalization of the method introduced by Filippov [1]; trajectories may exhibit sliding mode segments and higher-order non differentiable behavior as illustrated in Figure 1. In order to obtain existence of solutions,

multivalued extensions of f onto its discontinuity surfaces are required. Every trajectory can be represented by a sequence of transition points and times, $\{x(t_i), t_i\}$.

Definitions of various types of stability and instability can be constructed from an examination of the invariant limit sets [2] of the trajectories. For diced systems, the range of asymptotic behavior of trajectories starting from different initial conditions can be exceedingly rich; Figure 2 illustrates some patterns in R^2 . The possibility of approximate global stability analysis using nondeterministic automata is examined and its limitations are shown.

In practice, diced systems might be viewed as approximations of continuous or discontinuous systems. In the former case, for instance, we might seek the best piecewise-constant (finite-element) approximation to a continuous system. Wang [3] has presented an application of this type for solving partial differential equations. In the latter case, a state space diffeomorphism might be used first to transform the discontinuities of a system to lie along coordinate axes, and then a diced approximation could be developed which would preserve the discontinuous behavior of such systems. The potential practical advantages of diced approximations lie in a reduction of information storage required to characterize a system and the possibility of assessing its approximate asymptotic behavior without a detailed simulation.

For example, at the time of a known failure of a power system, it is often desirable to predict the long-term consequences of various control strategies so that an operator can decide among them. Yet the system is too

big to store all possible consequences in advance. A practice which has thus been followed in some cases [4] is to run a simulation "faster than real-time" for each control strategy. While the issue of approximation accuracy is not treated here, the results suggest that significant economy of real-time computation might be achieved by approximating the dynamics of a dived system. However, they also suggest that the patterns of stability and instability exhibited by such discontinuous systems may be highly complex and that analytical methods are not likely to yield clear-cut predictions about global stability.

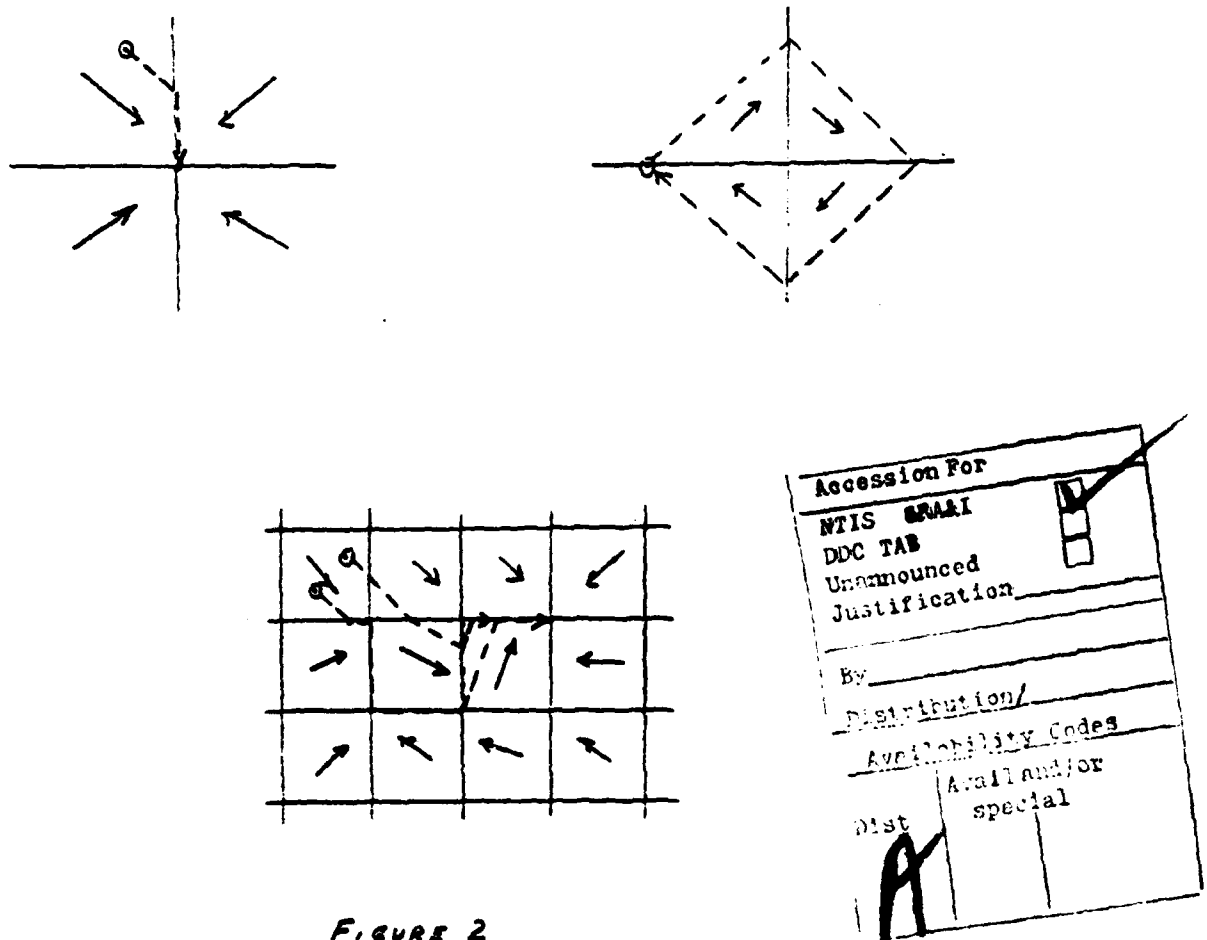


FIGURE 2

II. Preliminaries, Notation

Let $i = [i_1, \dots, i_n] \in \mathbb{Z}^n$ be a multi-index on the n-tuples of integers (Z). Let $b = [b_1, \dots, b_n] \in \mathbb{B}^n$ represent an n-tuple of binary numbers ($\mathbb{B} = \{0,1\}$). Let $\chi_i(x): \mathbb{R}^n \rightarrow \mathbb{R}$ be the characteristic function of the open set $\{x = [x_1, \dots, x_n] \in \mathbb{R}^n \mid i_k < x_k < i_k+1, k = 1, 2, \dots, n\}$.

Definition: A diced initial value problem (DIVP) is specified by a system of ordinary differential equations

$$\dot{x}(t) = f(x(t)) ; x(t_0) = x_0 \in \mathbb{R}^n ; t \geq t_0 \quad (2.1)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the particular form

$$f(x) = \sum_{i \in \mathbb{Z}^n} f_{bi} \chi_i(x) ; b = 0 \in \mathbb{B}^n \quad (2.2)$$

and $f_{bi} \in \mathbb{R}^n$ for each multi-index i .

The surfaces of discontinuity of f may be classified by their dimension. Let $\underline{1}(b): \mathbb{B}^n \rightarrow \{1, \dots, n\}$ be a function denoting the number of "1"'s in the binary n-tuple b . For fixed $i \in \mathbb{Z}^n$, consider the sets

$$S_{bi} = \{x \in \mathbb{R}^n \mid i_k < x_k < i_k+1 \text{ if } b_i = 0 \\ i_k = x_k \text{ if } b_k = 1, k = 1, 2, \dots, n\} \quad (2.3)$$

These may be viewed as the set of submanifolds "attached to" the point $x=i$.^{*} For example S_{0i} is the interior of the n-dimensional cube indexed by its vertex at $x=i$; S_{1i} (the shorthand 1 denoting $b = [1, 1, \dots, 1]$) is the single point $x=i$. The submanifolds of dimension p associated with $x=i$ are

^{*}The obvious injection of the integers into the reals is implied.

$$S_i^p = \{S_{bi} \mid l(b) = n-p\} \quad p = 0, 1, \dots, n. \quad (2.4)$$

This notation provides a compact classification of all of the subsets of R^n which are of interest.

In Section III, conditions for well-posedness of a DIVP are examined. This is done by extending f to its discontinuity surfaces (from $\{f_{oi}\}$, we generate $\{f_{bi}\}$, $b \neq 0 \in R^n$). Then a constructive procedure can be used to generate solutions $x(t) = \phi(t, t_0, x_0)$ for each $x_0 \in R^n$, $t_0 \in R$ and hence to define the transition map $\phi: R \times R \times R^n \rightarrow R^n$. Let X denote the function space in which trajectories are defined. This leads to the following

Definition: A diced system is an autonomous dynamical system (X, R^n, ϕ) (See. [5]).

Stability has been viewed as a qualitative property of a dynamical system, and concerns the asymptotic behaviors of trajectories $x(\cdot) = \phi(\cdot, t_0, x_0)$ as $x_0 \in X$ is varied. Stability of diced systems is discussed in Section IV. Two useful notions will be those of the positive limit set and the invariant set [2].

Definition: The set $\Omega \subset R^n$ is invariant with respect to the system $\dot{x}(t) = \tilde{f}(x(t), t)$ if for any $x_0 \in \Omega$ there is a t_0 such that the motion $\tilde{\phi}(t, t_0, x_0)$ belongs to Ω for all $t \geq t_0$.

Definition: The set $\Pi \subset R^n$ is called the positive limit set of a bounded motion $\phi(t; t_0, x_0)$ if, for any point $p \in \Pi$, there exists a sequence of times $\{t_n\}$ tending to infinity as $n \rightarrow \infty$, so that

$$\lim_{n \rightarrow \infty} \|\phi(t_n, t_0, x_0) - p\| = 0 \quad (2.5)$$

In applying these definitions it will be useful to recall that a function $x(t)$ is periodic $T > 0$ if $x(t) = x(t+T)$ for all t ; "the" period of a periodic function is defined as the least T for which this equality holds.

III. Existence and Uniqueness

Consider the DIVP (2.1), (2.2). Defining solutions within the cubes S_{oi} by integration is entirely straightforward; all difficulties arise in attempting to extend solutions across the discontinuity surfaces of f ; in general, there is no unique continuation. Various possibilities are

- (a) To restrict the class of f so that continuations are always unique (this is very restrictive indeed, and essentially eliminates many interesting phenomena from consideration).
- (b) To eliminate the non-continuable surfaces from the domain of f ; however, then all points on all trajectories leading to such surfaces must also be eliminated, and a large part of the original domain of definition may ultimately be excluded.
- (c) To choose an ad hoc rule for continuation of solutions; however, it proves difficult or impossible to do this in a self-consistent and unbiased manner.

A fourth alternative has been selected here:

- (d) To sacrifice uniqueness and continue all solutions through a discontinuity.

In this way a viable deterministic existence theory can be developed, at the cost of considering a countable number of alternative solutions. A "physical" justification for adopting this approach is that in the presence of small perturbations of the initial conditions, a solution near to at least one alternative solution will occur.

A constructive procedure is given for defining solutions. To simplify

its presentation, a multivalued continuation of f to the surfaces S_{bi} , $b \neq 0$, is first defined. Initially, f is specified on the submanifolds $S_i^n = \{S_{oi}\}$ of dimension n . The continuation proceeds recursively to submanifolds, S_i^p of dimension $n-1, n-2, \dots, 0$. Recall that S_i^0 is the point set $\{x \in \mathbb{R}^n \mid x = i_k, i_k \text{ an integer}\}$. Notationally, a single valued f_{bi} will not be distinguished from a multivalued f_{bi} , the implication being that the prescribed rule is applied to each possible value of f_{bi} in turn, and the set of all results is retained. Let $p = n$. Suppose f_{bi} are known on S_i^q , $p \leq q \leq n$. Then f_{bi} can be extended to S_i^{p-1} as follows, for each $i \in \mathbb{Z}^n$.

Suppose $S_{bi} \in S_i^{p-1}$. Let indices $j_1 \dots j_{n-(p-1)}$ denote the ordered nonzero positions of b , i.e., $b_{j_k} = 1, k = 1, \dots, n-(p-1)$ and $b_j = 0$ otherwise. The neighborhoods of S_{bi} of dimension q , $p \leq q \leq n$, can be defined as follows. For $q = n$, consider all indices \tilde{i} formed by decrementing i_{j_k} by one for any subset of the subindices $k = 1, \dots, n-(p-1)$, including the null-set; then $S_{\tilde{b}\tilde{i}} \in S_i^n$ is a neighborhood of S_{bi} where $\tilde{b} \neq 0$. For $q = n-1$, consider all values b having a single "one" in one of the positions $j_1 \dots j_{n-(p-1)}$ and for each \tilde{b} , form \tilde{i} from the remaining $n-(p-1)-1$ indices as above; then $S_{\tilde{b}\tilde{i}} \in S_i^{n-1}$ is a neighborhood of S_{bi} . For $q = n-2$, consider all values \tilde{b} having "ones" in any two of the positions $j_1, \dots, j_{n-(p-1)}$ and from each \tilde{b} form \tilde{i} from the remaining $n-(p-1)-2$ indices as above; then $S_{\tilde{b}\tilde{i}} \in S_i^{n-2}$ is a neighborhood of S_{bi} . This procedure is continued until $q = p$.

The values of f_{bi} on $S_{bi} \in S_i^{p-1}$ are determined from the values of $f_{\tilde{b}\tilde{i}}$ on each of its neighborhoods $S_{\tilde{b}\tilde{i}} \in S_i^q$, $p \leq q \leq n$. It is thus

sufficient to give the procedure for determining f_{bi} , assuming that these values on higher-dimensional submanifolds are known (i.e., the values can be determined recursively). Define S_{bi}^{\sim} to be an input submanifold to S_{bi} if $(f_{bi}^{\sim})_l = 0$ for all l such that $\tilde{b}_l = 1$, and for all remaining l in the set $j_1, \dots, j_{n-(p-1)}$, $(f_{bi}^{\sim})_l < 0$ for those l such that $\tilde{i}_l = i_l$, while $(f_{bi}^{\sim})_l > 0$ for those l such that $\tilde{i}_l = i_l - 1$. Define S_{bi}^{\sim} to be an output submanifold if $(f_{bi}^{\sim})_l = 0$ for all l such that $\tilde{b}_l = 1$, and for all remaining l in the set $j_1, \dots, j_{n-(p-1)}$, $(f_{bi}^{\sim})_l \geq 0$ for those l such that $\tilde{i}_l = i_l$, while $(f_{bi}^{\sim})_l \leq 0$ for those l such that $\tilde{i}_l = i_l - 1$. Note that those sets for which $(f_{bi}^{\sim})_l \neq 0$ when $\tilde{b}_l = 1$ need not be considered. So long as the set of output submanifolds of S_{bi} is non-empty, f_{bi} is assigned the set of all values f_{bi}^{\sim} on the output submanifolds. If the set of output submanifolds is empty, S_{bi} is a generalized sliding surface. Consider f_{bi} on $S_{bi} \in S_i^p$ in the input set. If this set is empty, set $f_{bi}^{\sim} = 0$. Recall that $S_{bi} \in S_i^p$ is formed by keeping i unchanged in all but one position, say j_k , of b , so $\tilde{b} = [b_1, \dots, b_{j_k-1}, 0, b_{j_k+1}, \dots, b_n]$ and either $\tilde{i} = i$ or $\tilde{i} = [i_1, \dots, i_{j_k-1}, i_{j_k+1}, \dots, i_n]$. Thus there are a maximum of $2(n-(p-1))$ surfaces in this subset of the input set. These surfaces are considered in pairs to determine the admissible values of f_{bi} ; using the example above, if S_{bi} is in the input set then $(f_{bi}^{\sim})_{j_k} < 0$ and if S_{bi}^{\sim} is in the input set $(f_{bi}^{\sim})_{j_k} > 0$. If both elements are members of the input set then

$$f_{bi} = [(f_{bi}^{\sim})_{j_k} f_{bi} - (f_{bi})_{j_k} f_{bi}^{\sim}] / [(f_{bi}^{\sim})_{j_k} - (f_{bi})_{j_k}] \quad (3.1)$$

while if only one is in the input set, let

$$f_{bi} = 0$$

The set of possible values of f_{bi} on a generalized sliding mode is completed by considering each $S_{bi} \in S_i^p$ in this manner. In all such cases, $(f_{bi})_{j_k}$; $k = 1, \dots, n-(p-1)$ are zero, so that further motion occurs on S_{bi} itself.

Thus, the procedure for extending the function f to all of R^n is completed. The complexity of the procedure arises from the large number of possibilities which can arise. A number of such special cases are illustrated on Figure 3. Evidently, the procedure for extending f is not the only one which could be devised. In the next step, construction of solutions, however, it will become apparent that the underlying principle has been to define f in a manner which preserves all trajectories that might arise from each initial condition.

Let $x_0 \in R^n$ be given as the initial condition of (2.1) at $t = t_0$; let $S_{bi} \in S_i^p$ be the smallest submanifold containing x_0 . Let f_{bi} denote one of the extended values of f on S_{bi} . Define

$$\phi(\tau, t_0, x_0) = x_0 + f_{bi}(\tau - t_0) ; \quad t_0 < \tau \leq t_1 \quad (3.2)$$

The time t_1 is defined as follows: for each ℓ such that $(f_{bi})_\ell$ is nonzero, let $(t_1)_\ell$ denote the first $\tau > t_0$ such that $[\phi(\tau, t_0, x_0)]_\ell$ is an integer; then $t_1 = \min_\ell [(t_1)_\ell]$ and $x_1 = \phi(t_1, t_0, x_0)$. If $f_{bi} = 0$, then $t_1 = \infty$ and $x_1 = x_0$, and this solution terminates. Otherwise, x_1 defines new values of b, i , and p , and the solution process continues:

$$\phi(\tau, t_k, x_k) = x_k + f_{bi}(\tau - t_k) ; \quad t_k < \tau \leq t_{k+1} \quad (3.3)$$

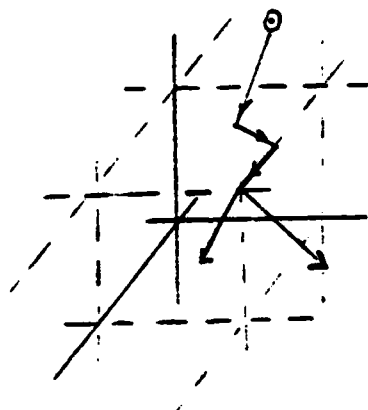
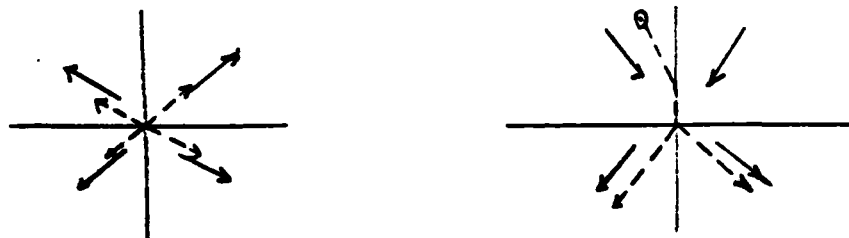


FIGURE 3

On those surfaces where f_{bi} is multivalued, each possibility must be examined in turn; in this sense, ϕ is also multivalued. Each trajectory pieced-together in this fashion can be summarized by a sequence $\{x_k, t_k\}$, $k = 0, 1, \dots$ in some cases, these sequences are finite and in other cases infinite. By inspection of $\{x_k\}$ alone, a corresponding sequence of regions $\{\sigma_k\}$, where $\sigma_k \in \{S_{bi}\}$ is the minimal submanifold containing x_k , can be constructed.

A solution of (2.1), (2.2) is then defined in the obvious manner, as any $\phi(t, t_0, x_0)$ constructed by the continuation procedure (3.3). It has the property that for any finite admissible k , $\phi(t, t_0, x_0)$ is piecewise continuous on $[t_0, t_k]$. This solution by continuation is said to be asymptotic if $\lim_k t_k = \infty$. An asymptotic solution is piecewise continuous. For purposes of the present work, a solution will be said to exist if the state-space continuation is asymptotic.* Asymptotic solutions need not be unique, but the rate of growth in the number of solutions can be bounded as a function of k , since the maximum number of output submanifolds can be bounded above for any S_{bi} . If there is only one asymptotic solution through (x_0, t_0) , it is said to be unique. Continuous dependence of $\phi(t, t_0, x_0)$ with respect to x_0 , of course, is not to be expected for $t > t_1$.

*Moreover if $\lim_k t_k \rightarrow \infty$, solutions by time-continuation could be defined; however, their properties will not be explored here.

IV. Stability

The usual definitions of stability presuppose a solution which is well-posed in the sense of existence, uniqueness, and continuous dependence on the initial data. Diced systems, in general, do not possess the last two properties. One alternative is to nevertheless use the standard notions of stability, restricting their domain of application to those initial states for which the usual notions of well-posedness are (locally) satisfied. Unfortunately, the set of such initial states appears quite difficult to characterize and thus imposes an awkward restriction on the applicability of this alternative.

Another alternative, introduced here, does not impose such restrictions, but weakens the notion of stability that is employed. Stability is viewed as a qualitative property of a trajectory, and a system is then said to be stable when all of its trajectories share this property.

Definition: The motion of a diced system (2.1), (2.2) initiated at (t_0, x_0) is

$$M(t_0, x_0) = \{ \phi(t, t_0, x_0), t \geq t_0 \mid \phi \text{ is a transition function} \\ \text{initiated at } (t_0, x_0) \}$$

which is the set of all trajectories originating at (t_0, x_0) .

Definition: The motion $M(t_0, x_0)$ of a diced system is said to be

(a) Bounded in magnitude if there is a constant $\phi > 0$ such that

$$\max_{\phi \in M(t_0, x_0)} \left\{ \sup_{t \geq t_0} ||\phi(t, t_0, x_0)|| \right\} < \phi$$

(b) Bounded in cardinality if there exists a constant N such that

$$\sup_{t \geq t_0} \{\text{cardinality of } \phi(t, t_0, x_0)\} < N$$

The concepts of boundedness in magnitude and cardinality are independent. In both cases, the only difficulties occur at $t \rightarrow \infty$, since (a) any $\phi(t, t_0, x_0)$ is by construction bounded for all finite t , and (b) the cardinality of $\phi(t, t_0, x_0)$ is finite, by construction, for all finite t . The following propositions are almost immediate.

Proposition 1: In (2.1), suppose $\|f_{oi}\| < F$ for all i , then $\|\phi(t, t_0, x_0) - x_0\| < F(t - t_0)$ for all $\phi \in M(t_0, x_0)$.

Proof: The extension of f_{oi} to f_{bi} always guaranteed that $\|f_{bi}\| < F$, and the construction procedure (3.3) guaranteed that the estimate of the proposition held for each t . q.e.d.

Proposition 2: Let $|i| = |i_1| + \dots + |i_n|$. Suppose for system (2.1) there exists $B > 0$ such that for all $|i| > B$, and $k = 1, \dots, n$, $(f_{oi})_k i_k < 0$. Then $M(t_0, x_0)$ is bounded in magnitude.

Proof: For any i such that $|i| > B$, every set S_{bi} contains output submanifolds with the same $|i|$ or smaller $|i|$, and input submanifolds with the same $|i|$ or larger $|i|$; furthermore, S_{1i} always outputs to S_{oi} with $|i| < |i|$. Thus the construction process cannot terminate for $|i| > B$, and for such i , $|i|$ is reduced at least once every n intervals; hence every solution satisfies $|\phi(t, t_0, x_0)| < B$ for t sufficiently large. Thus $M(t_0, x_0)$ is magnitude-bounded.

Proposition 3: Suppose that for every $i \in Z^n$, $b \in B^n$, S_{bi} has at most one output submanifold. Then the motion $M(t_0, x_0)$ of (2.1), (2.2) is bounded in cardinality.

Proof: The extension procedure of Section III shows that in this case f_{bi} takes the value on its output submanifold or the value zero. If a trajectory enters S_{bi} , it either continues uniquely to the output submanifold, or terminates at S_{bi} . In either case, the cardinality of the solution cannot increase during its construction.

Thus there are two notions of instability for diced systems: solutions may become unbounded in magnitude, and/or they may become unbounded in cardinality. This second form of instability is new: a trajectory can fracture and a chain reaction of subsequent fractures may ensure--the complexity of the process grows without bound.

Next, a notion of stability is put forth. Suppose that the motion $M(t_0, x_0)$ of a diced system is bounded in magnitude and cardinality (or simply "bounded"). Then a set $S \subset R^n$ consisting of a finite union of the submanifolds S_{bi} is termed a positive limit set of a (bounded) trajectory $\phi(t, t_0, x_0)$ if for any point $x \in S$, there exists a sequence of times $\{\tau_k\}$, tending to infinity as $k \rightarrow \infty$, so that

$$\lim_{k \rightarrow \infty})\phi(\tau_k, t_0, x_0) - x(= 0 \quad (4.1)$$

where $)$ (denotes the set-membership metric, i.e., if $x \in S_{bi}$,

$$)y-x(= \begin{cases} 0 & y \in S_{bi} \\ 1 & y \notin S_{bi} \end{cases}$$

In applying this definition, it is important to recall the standing assumption from Section III, that all trajectories are asymptotic, so that such sequences $\{\tau_k\}$ exist.

Definition: A bounded motion $M(t_0, x_0)$ of a diced system is termed pointwise stable if all trajectories $\phi(t, t_0, x_0) \in M(t_0, x_0)$ have the same positive limit set. The motion is locally stable for $x_0 \in S_{bi}$ if all trajectories $\phi(t, t_0, x) \in M(t_0, x)$, $x \in S_{bi}$, have the same positive limit set. The motion is globally stable if all trajectories $\phi(t, t_0, x)$ have the same positive limit set.

Concepts of uniform stability will not be discussed since only time-invariant diced systems are considered in the present account.* In fact, the evaluation of stability, according to the definitions given, can be based merely on knowledge of the sequence $\{\sigma_k\}$ of submanifolds containing $\{x_k\}$, since it is known from the construction procedure that $t_{k+1} > t_k$ and from the asymptotic assumption that $\lim_{k \rightarrow \infty} t_k = \infty$. This suggests that a way to generate the sequence $\{\sigma_k\}$ autonomously, without explicit integration and generation of $\{x_k, t_k\}$ would be particularly valuable in the assessment of stability. This has not been achieved yet.

Knowledge of the time-structure $\{t_k\}$ of individual solutions can be of further value in refining stability notions. To simplify the remaining concepts it is now assumed that the trajectories are uniquely-defined

*The results could be extended in this direction for systems with continuous time-variation; however discontinuously time-varying systems may not be continuous, as Filippov pointed out.

(e.g., as occurs in Proposition 3) and bounded. Suppose Π is a positive limit set of such a solution in the conventional sense of Section II (eq. (2.5)). Then in the usual manner it can be shown that Π is bounded, closed, non-empty and invariant, the last property being a consequence of time-invariance. In fact, as a consequence of finite-dimensionality of R^n , all such solutions are asymptotically almost-periodic [6]. Two cases of special interest are the asymptotically constant (equilibrium) solution and the asymptotically periodic solution. These can be identified directly from the sequence $\{x_k, t_k\}$ characterizing $\phi(t, t_0, x_0)$.

Proposition 4: If the sequence $\{x_k, t_k\}$ is finite of length N , the positive-invariant limit set consists of one point, the last value x_N (for which $t_N = \infty$). If the sequence $\{x_k, t_k\}$ is jointly periodic of period m for $k > N$, then the positive-invariant limit set is a cycle (closed curve) in R^n .

Proof: For the first case, note that the construction procedure automatically defines $t_N = \infty$ when the sequence is finite, and this implies a constant solution for $t \geq t_N$. In the second case, note that since $\{x_k, t_k\}$ completely specify $\phi(t, t_0, x_0)$, ϕ must be periodic of period $t_{k+m} - t_k$, $k > N$, whenever $\{x_k, t_k\}$ is periodic (in fact, the solution is a linear interpolation between these points).

It is interesting to note that for diced systems, the establishment of an equilibrium or periodic solution after a finite time (t_N) is often to be expected (whereas this would be considered exceptional in the case of continuous differential equations); however, in some cases almost periodic solutions may also exist.

V. Discussion and Conclusions

The present account of the stability of diced systems leaves a number of original question unanswered and raises some new ones. A study of methods for temporal continuation of non-asymptotic solutions is needed; such solutions may represent a new sort of sliding mode which can arise in higher dimensional spaces, as suggested by an example of Utkin [7]. The possibility of extending the techniques developed here to time-varying systems has been mentioned; Filippov's general existence results apply to this problem. A study of the partitioning of initial states which is implied by the proposed stability definition would also be fruitful; what properties are shared by initial state sets giving rise to the same asymptotic solution? In general, it would appear that the initial states within a given region S_{bi} can ultimately end up widely dispersed. The possibility of using an automaton to simplify the propagation of solutions has also been raised. The approximation of continuous systems by diced systems has not been explored, but under appropriate conditions, a bound on the approximation error should be achievable.

In spite of the questions that are unanswered, some modest progress has been made toward defining the stability properties of diced systems. First, a constructive continuation procedure for higher dimensions has been found; the problem readily evades one's intuition about $n = 1, 2$ and even 3 as endless combinations of difficult situations may occur. Second, a compromise on the issue of uniqueness has been put forth: the number of admissible solutions at any finite time is bounded. Third, the concepts of stability have been generalized to provide meaningful criteria for discon-

tinuous systems of diced type.

Returning to the electric power system example cited in the opening section, it would appear that the implications of the research might be very disturbing, for two primary reasons. First, a new type of instability--an unbounded growth in the number of possible solutions with time--has been identified. Second, and independently, the partitioning of the initial state--at least in worst-case situations--based on asymptotic properties, appears to be very fine and irregular; thus a small perturbation in the initial state may give rise to completely different asymptotic behavior than is found for the unperturbed initial state. Both of these phenomena imply that the future behavior of a diced system with a (approximately) specified initial state may be fundamentally unpredictable; if the long-term future consequences of a present control policy are unpredictable, the problem of choosing the best policy becomes more difficult and planning must be done with a shorter horizon.

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